14 Linear Programming

INTRODUCTION

You are familiar with linear equations and linear inequations in one and two variables. They can be solved algebraically or graphically (by drawing a line diagram in case of one variable, and on two dimensional graph paper in case of two variables x and y). We already know how to solve a system of linear inequations and we will utilise this knowledge extensively in solving problems on linear programming.

LINEAR PROGRAMMING

In a military operation, the effort is to inflict maximum damage to enemy at minimum cost and loss. In an industry, the management always tries to utilise its resources in the best possible manner. The industrialist would like to have unlimited profits, but he is constrained by limited manpower, capital and market demand. A salaried person tries to make investments in such a manner that the returns on investment are high but at the same time income tax liability is also kept low.

In all the above cases, if constraints are represented by linear equations/inequations (in one, two or more variables), and a particular plan of action from several alternatives is to be chosen, we use *linear programming*. The word *linear* means that all inequations used and the function to be maximized or minimized are linear, and the word *programming* refers to *planning* (choosing amongst alternatives) rather than the computer programming sense. American economist Dantzig is credited with developing the subject of linear programming.

Thus, linear programming is a method for determining optimum values of a linear function subject to constraints expressed as linear equations or inequalities.

A practical problem may involve dozens of variables, and is usually solved by using Simplex Method and a computer. In this chapter, we will restrict ourselves to linear functions involving two variables, so that they can be solved by drawing graph on xy-plane.

Three Classical Linear Programming Problems (L.P.P.)

(*i*) *Manufacturing problems*

When a firm has the choice of manufacturing a mix of different products, and each product requires a fixed manpower, machine hours, labour hours per unit of the product, warehouse space per unit of the output etc., it has to choose which products should be produced in what numbers so as to maximize the profits.

(*ii*) *Diet problems*

Different kinds of foods have different amounts of nutrients (vitamins, minerals etc.). In diet problems, we have to determine the amounts of different kinds of foods which should be included in the diet so as to minimize the cost of the diet such that it contains a certain (minimum) amount of each nutrient.

(*v*) Both the objective function and constraints must be expressed in terms of *linear* equations or inequations.

(*vi*) Programming/planning requires that there must be *alternative courses of line of action i.e.* there are alternatives available to which resources can be put.

(*vii*) All decision variables should assume non-negative values.

Advantages of Linear Programming

We have already mentioned a few application areas of linear programming-military operations, choosing optimal product line, diet problems and transportation problems. It helps in choosing the best alternative from a set of feasible alternatives, so that profit is maximized, or costs are minimized. Some other application areas are :

(*i*) Choosing the media mix (radio, TV, newspapers, hoardings, magazines, internet) to maximize the advertising effectiveness, within given publicity budget.

(*ii*) Determining shortest routes for travelling salesmen.

(*iii*) Helping a farmer decide best crop mix so as to minimize risk and maximize profit.

Thus, the linear programming helps in

(*i*) choosing the best alternative amongst a set of alternatives so that profit is maximized or cost is minimized.

(*ii*) taking into consideration not only internal factors like manpower, machine, budget, storage availability etc. but also external factors like market demand, purchasing power of the customer etc.

(*iii*) identifying bottlenecks in the production process.

(*iv*) choosing best production policy and inventory policy so that seasonal fluctuations in demand can be handled.

Limitations of Linear Programming

(*i*) It deals with optimizing a single objective. In practice, a number of objectives may be there.

(*ii*) The assumption that input and output variables are directly proportional is not strictly true. *Economies of scale* usually ensure that the more you produce, lesser is the average cost.

(*iii*) The linearity of variables assumes that resources required for multiple activities are sum total of resources required for individual activities. However, *synergies* of product mix usually mean that the requirement is less than the sum.

(*iv*) In practice, many decision variables assume *integral values, e.g.,* number of workers. L.P. deals with variables having continuous values.

FORMULATION OF AN L.P.P. IN TWO VARIABLES x AND y

There are *three* steps in the mathematical formulation of an L.P.P. and solving it. These are:

(*i*) To identify the objective function as a linear combination of variables (x and y) and to construct all constraints *i.e.* linear equations and inequations involving these variables. Thus, an L.P.P. can be stated mathematically as

Maximize (or minimize) $Z = ax + by$

subject to the conditions

$a_i x + b_i y \leq (or \geq or = or > or <)\ c_i$, *where* $i = 1$ *to* n

and the non-negative restrictions

$x \geq 0, y \geq 0.$

(*ii*) To find the solutions (*feasible region*) of these equations and inequations by some mathematical method. Here we will study only the graphical method.

(iii) To find the *optimal solution* i.e. to select particular values of the variables x and y that give the desired value (maximum/minimum) of the objective function.

ILLUSTRATIVE EXAMPLES

Example 1. *A shopkeeper deals in two items—wall hangings and artificial plants. He has ₹15000 to invest and a space to store atmost 80 pieces. A wall hanging costs him ₹300 and an artificial plant ₹ 150. He can sell a wall hanging at a profit of ₹50 and an artificial plant at a profit of ₹18. Assuming that he can sell all the items that he buys, formulate a linear programming problem in order to maximize his profit.*

Solution. Let x be the number of wall hangings and y be the number of artificial plants that the dealer buys and sells. Then the profit of the dealer is $Z = 50x + 18y$, which is the objective function.

As a wall hanging costs ₹300 and an artificial plant costs ₹150, the cost of x wall hangings and y artificial plants is $300x + 150y$. We are given that the dealer can invest atmost ₹15000. Hence, the investment constraint is

$$300x + 150y \leq 15000 \text{ i.e. } 2x + y \leq 100$$

As the dealer has space to store atmost 80 pieces, we have another constraint (space constraint):

$$x + y \leq 80$$

Also the number of wall hangings and artificial plants can't be negative. Thus we have the non-negativity constraints :

$$x \geq 0, y \geq 0.$$

Thus the mathematical formulation of the L.P.P. is :

Maximize $Z = 50x + 18y$ *subject to the constraints*

$$2x + y \leq 100,\ x + y \leq 80,\ x \geq 0,\ y \geq 0.$$

Example 2. *A retired person wants to invest an amount of upto ₹20000. His broker recommends investing in two types of bonds A and B, bond A yielding 8% return on the amount invested and bond B yielding 7% return on the amount invested. After some consideration, he decides to invest atleast ₹5000 in bond A and no more than ₹8000 in bond B. He also wants to invest atleast as much in bond A as in bond B. Formulate as L.P.P. to maximize his return on investments.*

Solution. Let the person invest ₹x in bonds of type A and ₹y in bonds of type B, then his earning *i.e.* return (in ₹)

$$Z = 8\% \text{ of } x + 7\% \text{ of } y = 0.08x + 0.07y.$$

As the person can invest upto ₹20000, so investment constraint is

$$x + y \leq 20000.$$

Other investment constraints are

$$x \geq 5000, y \leq 8000, x \geq y.$$

Non-negativity constraints are $x \geq 0, y \geq 0$.

Thus, mathematically formulation of the L.P.P. is

Maximize $Z = 0.08x + 0.07y$ subject to the constraints

$$x + y \leq 20000, x \geq 5000, y \leq 8000, x \geq y, x \geq 0, y \geq 0.$$

Example 3. (*Manufacturing problem*) *A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts, while it takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹2.50 per package on nuts and ₹1 per package on bolts. Form a linear programming problem to maximize his profit, if he operates each machine for atmost 12 hours.*

Solution. Suppose that x packages of nuts and y packages of bolts are produced. The objective of the manufacturer is to maximize the profit $Z = 2.50x + 1.y$

Time required on machine A to produce x packages of nuts and y packages of bolts is $1.x + 3.y = x + 3y$, while time required on machine B is $3.x + 1.y = 3x + y$.

From given data, we can formulate the L.P.P. as :

Maximize $Z = 2.50x + y$ *subject to the constraints*

$x + 3y \leq 12$ (*Machine A constraint*)

$3x + y \leq 12$ (*Machine B constraint*)

$x \geq 0, y \geq 0$ (*Non-negativity constraints*)

Example 4. *A manufacturer has three machines M_1, M_2 and M_3 installed in his factory. Machines M_1 and M_2 are capable of being operated for atmost 12 hours, whereas machine M_3 must be operated for atleast 5 hours a day. The manufacturer produces only two items, each requiring the use of these three machines. The following table gives the number of hours required on these machines for producing 1 unit of A or B.*

Item	*Number of hours required on the machines*		
	M_1	M_2	M_3
A	1	2	1
B	2	1	$\frac{5}{4}$

He makes a profit of ₹60 on item A and ₹40 on item B. He wishes to find out how many of each item he should produce to have maximum profit. Formulate it as L.P.P.

Solution. Let the manufacturer produce x units of item A and y units of item B. As he makes a profit of ₹60 on one item of A and ₹40 on one item of B, his total profit (in ₹) is

$$Z = 60x + 40y.$$

From the given data, we can mathematically formulate the L.P.P. as

Maximize $Z = 60x + 40y$ subject to the constraints

$x + 2y \leq 12$ (Machine M_1 constraint)

$2x + y \leq 12$ (Machine M_2 constraint)

$x + \frac{5}{4}y \geq 5$ (Machine M_3 constraint)

$x \geq 0,\ y \geq 0$ (Non-negativity constraints)

Example 5. (*Diet problem*) *A diet is to contain atleast 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs ₹4 per kg and F_2 costs ₹5 per kg. One kg of food F_1 contains 3 units of vitamin A and 4 units of minerals. One kg of food F_2 contains 6 units of vitamin A and 3 units of minerals. We wish to find the minimum cost for diet that consists of mixture of these two foods and also meets the minimum nutritional requirements. Formulate this as a linear programming problem.*

Solution. Let the mixture consist of x kg of food F_1 and y kg of food F_2.

We make the following nutritional requirement table from given data :

Resources	*Food (in kg)*		*Requirement (in units)*
	F_1	F_2	
Vitamin A (units/kg)	3	6	80
Minerals (units/kg)	4	3	100

Minimum requirement of vitamin A is 80 units,

therefore, $3x + 6y \geq 80$

Similarly, the minimum requirement of minerals is 100 units,

therefore, $4x + 3y \geq 100$

Also $x \geq 0,\ y \geq 0$

As cost of food F_1 is ₹4 per kg, and the cost of food F_2 is ₹5 per kg, the total cost of purchasing x kg of food F_1 and y kg of food F_2 is $Z = 4x + 5y$, which is the objective function.

Hence, the mathematical formulation of the L.P.P. is :

Minimize $Z = 4x + 5y$ subject to the constraints

$3x + 6y \geq 80,\ \ 4x + 3y \geq 100,\ \ x \geq 0,\ y \geq 0.$

Example 6. *(Transportation problem) A brick manufacturer has two depots A and B, with stocks of 30000 and 20000 bricks respectively. He receives orders from three builders P, Q and R for 15000, 20000 and 15000 bricks respectively. The cost (in ₹) of transporting 1000 bricks to the builders from the depots are given below :*

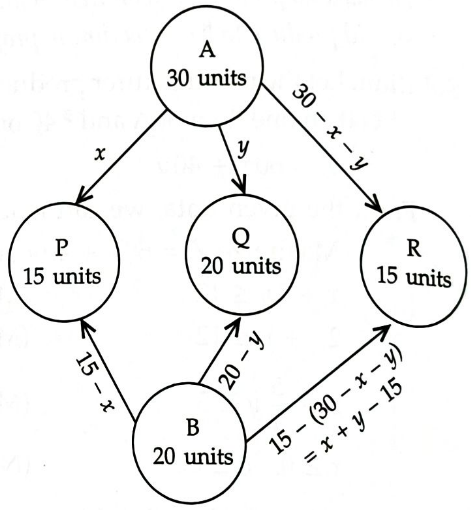
To / From	Transportation cost per 1000 bricks (in ₹)		
	P	Q	R
A	40	20	20
B	20	60	40

The manufacturer wishes to find how to fulfil the order so that transportation cost is minimum. Formulate the L.P.P.

Solution. To simplify, assume that 1 unit = 1000 bricks

Suppose that depot A supplies x units to P and y units to Q, so that depot A supplies $(30 - x - y)$ units bricks to builder R.

Now as P requires a total of 15000 bricks, it requires $(15 - x)$ units from depot B. Similarly Q requires $(20 - y)$ units from B and R requires $15 - (30 - x - y) = (x + y - 15)$ units from B.



Using the transportation cost given in table, total transportation cost is

$$Z_1 = 40x + 20y + 20(30 - x - y) + 20(15 - x) + 60(20 - y) + 40(x + y - 15)$$
$$= 40x - 20y + 1500$$

Obviously the constraints are that all quantities of bricks supplied from A and B to P, Q, R are non-negative,

i.e. $x \geq 0,\ y \geq 0,\ \ 30 - x - y \geq 0,\ \ 15 - x \geq 0,\ \ 20 - y \geq 0,\ \ x + y - 15 \geq 0$

Instead of minimizing $Z_1 = 40x - 20y + 1500$, it is easier to minimize

$Z = 40x - 20y$.

Hence the problem can be formulated as L.P.P. as

Minimize $Z = 40x - 20y$ *subject to the constraints*

$x + y \geq 15,\ x + y \leq 30,\ \ x \leq 15,\ y \leq 20,\ \ x \geq 0,\ y \geq 0.$

EXERCISE 14.1

1. Two tailors, A and B, earn ₹300 and ₹400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce atleast 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.

2. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is atmost 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹100 and that on a bracelet is ₹300, formulate an L.P.P. for finding how many of each should be produced daily to maximize the profit? It being given that atleast one of each must be produced.

3. A furniture dealer deals in only two items—tables and chairs. He has ₹20000 to invest and a space to store atmost 80 pieces. A table costs him ₹800 and a chair costs him ₹200. He can sell a table for ₹950 and a chair for ₹ 280. Assume that he can sell all the items that he buys. Formulate this problem as an L.P.P. so that he can maximize his profit.

4. A book publisher sells a hard cover edition of a book for ₹72 and a paperback edition for ₹40. In addition to a fixed weekly cost of ₹9600, the cost of printing hard cover and paperback editions are ₹56 and ₹28 per book respectively. Each edition requires 5 minutes on the printing machine whereas hardcover binding takes 10 minutes and paperback takes 2 minutes on the binding machine. The printing machine and the binding machine are available for 80 hours each week. Formulate the linear programming problem to maximize the publisher's profit.

Answers

1. Minimize $Z = 300x + 400y$, subject to the constraints

   $3x + 5y \geq 30, \quad x + y \geq 8, \quad x \geq 0, \quad y \geq 0$

2. Maximize $Z = 100x + 300y$, subject to the constraints

   $x + 2y \leq 32, \quad x + y \leq 24, \quad x \geq 1, \quad y \geq 1$

3. Maximize $Z = 150x + 80y$, subject to the constraints

   $4x + y \leq 100, x + y \leq 80, x \geq 0, y \geq 0$

4. Maximize profit $Z = 16x + 12y - 9600$

   Subject to the constraints

   $x + y \leq 960, 5x + y \leq 2400, x, y \geq 0.$